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Interpolation by Quadratic Splines

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Convergence properties of quadratic spline interpolation of continuous functions that does not necessarily take place at the midpoints of mesh intervals are investigated. A theorem giving lower bounds on the elements of the inverse of certain tridiagonal matrices is proved. This result is used to precisely relate the norm of certain interpolating projections to the points of interpolation and local mesh ratios. It is shown, for example, that for Lipschitz continuous functions, any choice of interpolation points, one in each mesh interval, uniformly bounded away from the mesh points, yields convergence at the best possible rate with no mesh ratio restriction.

1. INTRODUCTION

The purpose here is to study the convergence properties of quadratic splines interpolating a given continuous function. Particular attention is paid to the relation between the choice of interpolation points and the boundedness in $C[0, 1]$ of the resulting projection. Marsden [6] showed that quadratic spline interpolation at the midpoints of mesh intervals gives rise to projections that are uniformly bounded in $C[0, 1]$. In [3], Kammerer *et al.* extended Marsden's result by proving, among other things, convergence of derivatives and a local convergence theorem. They also applied their results to the numerical solution of a two-point boundary value problem. It is natural to expect that interpolation at points "close" to the midpoints of the mesh intervals would yield results similar to those just mentioned. One of our purposes is to make the word "close" as precise as possible. In Section 3 we obtain *sharp* bounds on how far away from the midpoints one may interpolate and still have uniformly bounded projections. One result, Theorem 3.2, is valid for all partitions and any choice of interpolation points satisfying a certain inequality. Another result, Theorem 3.4, relates a *local* mesh ratio restriction to the boundedness of projections determined by a single parameter λ. In Section 4, we show that for the class of Lipschitz continuous

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functions quadratic spline interpolation always converges at the correct rate as long as the points of interpolation are kept uniformly away from the mesh points. This is somewhat similar to the easily proved result that continuous piecewise linear interpolation always converges to a given continuous function as long as the points of interpolation are kept uniformly away from the midpoints of the mesh intervals. Since de Boor [1] has investigated quadratic spline interpolation at the mesh points, this paper may be thought of as interpolating the results of [3] (or [6]) and [1]. We note that Sharma and Tzimbalario [9] have also considered extensions of Marsden's results.

The methods used are matrix theoretic. All interpolation problems under consideration here give rise to nonnegative tridiagonal matrices. Using the fact that such matrices can be symmetrized [8], we prove bounds for the elements of the inverse of such a matrix. This result, which is of some interest in itself, complements a result of Kershaw [4]. Of course it was motivated by [4]. These results are contained in Section 2.

Finally, we note that it is possible to apply our results, at lest in theory, to the numerical solution of some operator equations. However, we do not do this since, by now, such applications are standard (see also the remarks at the end of the paper).

The notation in this paper is fairly standard and can be found in [3, 10].

2. Bounding the Elements of the Inverse of Certain Tridiagonal Matrices

If $M = (m_{ij})$ is a tridiagonal $N \times N$ matrix, the notation $M = (a_i\, b_i\, c_i)$ means that $m_{i,i-1} = a_i$, $m_{i,i} = b_i$, $m_{i,i+1} = c_i$. We leave a_1 and c_N undefined. We are interested in estimating the elements of $A^{-1} = (\alpha_{ij})$, where $A = (a_i\, 1\, c_i)$, $a_{i+1} c_i > 0$, $1 \leqslant i \leqslant N-1$, and $|a_i| + |c_i| \leqslant 1\ \forall i$. Kershaw [4] has obtained upper bounds on the $|\alpha_{i,j}|$'s which show that for the case $|a_i| + |c_i| \leqslant \gamma < 1\ \forall i$, the α_{ij}'s decay like $r^{|i-j|}$ for some $0 < r < 1$, r depending only on γ. Using different methods, we shall obtain *lower bounds* for the $|\alpha_{ij}|$'s. We first reduce the problem to the case of A a symmetric matrix, cf. [8, p. 157].

PROPOSITION 2.1. *Let $A = (a_i\, 1\, c_i)$ be an $N \times N$ matrix with $a_{i+1} c_i > 0$, $1 \leqslant i \leqslant N-1$. Then, there is a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_N)$ such that*

$$\tilde{A} = DAD^{-1} = (r_i\, 1\, r_{i+1}) \qquad \text{where } r_i = (a_i c_{i-1})^{1/2},\ 2 \leqslant i \leqslant N.$$

Proof. Let $d_1 = 1$, $d_j = d_{j-1}(c_{j-1}/a_j)^{1/2}$, $2 \leqslant j \leqslant N$.
Note that if $A^{-1} = (\alpha_{ij})$ and $\tilde{A}^{-1} = (\tilde{\alpha}_{ij})$, then $\alpha_{ij} = d_j^{-1} d_i \tilde{\alpha}_{ij}$.

We now assume that $A = (r_i\ 1\ r_{i+1})$ and $A^{-1} = (\alpha_{ij})$. Let e_j be the usual jth coordinate vector in $\mathbb{R}^N$. The jth column of A^{-1} is the solution of $Ax = e_j$. Define $\hat{A} = (-r_i\ 1\ -r_{i+1})$ and note that if $\hat{A}y = e_j$ then the coordinates of y are related to those of x by $y_j = (-1)^{j+1} x_j$, $1 \leqslant j \leqslant N$. Writing $A = I - B$, we have $y = (I - B)^{-1} e_j = \sum_{m=0}^{\infty} B^m e_j$ if $\rho(B) < 1$, where $\rho(B)$ is the spectral radius of B. Since the components of $B^m e_j$ are all nonnegative, we have that

$$y_i > (B^{i-j} e_j)_i \text{ if } i \geqslant j$$

and

$$y_i > (B^{j-i} e_j)_i \text{ if } i < j.$$

Consequently, since $B = (r_i\ 0\ r_{i+1})$, we have

$$| x_j | = y_j > 1,$$

$$| x_i | = y_i > \prod_{k=j+1}^{i} r_k, \qquad \text{if } i > j,$$

and

$$| x_i | = y_i > \prod_{k=i+1}^{j} r_k, \qquad \text{if } i < j.$$

LEMMA 2.2. *Assume that $r_i > 0$ for all i and that $\rho(I - A) < 1$. Then, with $A^{-1} = (\alpha_{ij})$ we have*

$$| \alpha_{ij} | > \prod_{k=m}^{n} r_k, \tag{2.1}$$

where $m = j + 1$ and $n = i$ if $i \geqslant j$ and $m = i + 1$, $n = j$ if $i < j$ (with the convention that $\prod_{k=j+1}^{j} r_k = 1$).

THEOREM 2.3. *Let $A = (a_i\ 1\ c_i)$ be a tridiagonal $N \times N$ matrix with $a_i c_{i-1} > 0$, $2 \leqslant i \leqslant N$. Assume that $\rho(A - I) < 1$. Then, with $A^{-1} = (\alpha_{ij})$ we have*

$$\prod_{k=i+1}^{j} | a_k | < | \alpha_{ij} |, \qquad \text{if } i < j, \tag{2.2}$$

and

$$\prod_{k=j+1}^{i} | c_{k-1} | < | \alpha_{ij} |, \qquad \text{if } i > j. \tag{2.3}$$

Proof. Combine Proposition 2.1 and Lemma 2.2.

3. Approximation of Continuous Functions

In this section we study the convergence properties of quadratic spline interpolation of *continuous* functions. Let $\Delta: 0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of [0, 1]. For a vector $\bar{\lambda} = (\lambda_1, ..., \lambda_N)$, $0 < \lambda_i < 1$, we consider the projection $P(= P(\bar{\lambda}, \Delta))$: $C[0, 1] \to S(2, \Delta) \equiv \{f \in C^1[0, 1]: f$ is a quadratic polynomial on each $[x_i, x_{i+1}]$, $0 \leqslant i \leqslant N-1\}$, defined by the conditions $P(\bar{\lambda}, \Delta) f = s$ if and only if $0 = (s-f)(0) = (s-f)(1) = (s-f)(\xi_i)$, $1 \leqslant i \leqslant N$, where $\xi_i = \lambda_i x_{i-1} + (1-\lambda_i) x_i$. For $\sigma \geqslant 1$, let $\mathscr{P}_\sigma = \{\Delta: 1/\sigma \leqslant (x_{i+1} - x_i)/(x_i - x_{i-1}) \leqslant \sigma, 1 \leqslant i \leqslant N-1\}$. With $h_i = x_i - x_{i-1}$, it is easy to check that $P(\bar{\lambda}, \Delta) f = s$ if and only if for $1 \leqslant i \leqslant N-1$

$$a_i s_{i-1} + s_i + c_i s_{i+1} = d_i^{-1}\{h_{i+1}(1-\lambda_{i+1})(1-\lambda_i)^{-1} f(\xi_i) + h_i \lambda_i \lambda_{i+1}^{-1} f(\xi_{i+1})\}, \tag{3.1}$$

where

$$d_i = h_{i+1}(1-\lambda_{i+1})(1+\lambda_i) + h_i\lambda_i(2-\lambda_{i+1}),$$

$$a_i = \lambda_i^2(1-\lambda_{i+1})\, h_{i+1}(1-\lambda_i)^{-1} d_i^{-1},$$

and

$$c_i = (1-\lambda_{i+1})^2\, \lambda_i h_i \lambda_{i+1}^{-1} d_i^{-1}.$$

Since $h_{i+1}(1-\lambda_{i+1})\, d_i^{-1} < (1+\lambda_i)^{-1}$ and $h_i\lambda_i d_i^{-1} < (2-\lambda_{i+1})^{-1}$, the right-hand side of (3.1) can be bounded above by $\| f \|_\infty (1-\lambda_i^2)^{-1} (2\lambda_{i+1} - \lambda_{i+1}^2)^{-1}$. Also, if $Pf = s$, then $\max_{x_{i-1} \leqslant x \leqslant x_i} | s(x) | \leqslant K(\lambda_i)\{| s_i | + | s_{i-1} | + | f(\xi_i)|\}$, where $K(\lambda_i)$ is a constant depending only on λ_i. Therefore, $\| P \|$ can be bounded above in terms of the quantities $\min_i \lambda_i$, $\min_i \{1-\lambda_i\}$, and $\| A^{-1} \|_\infty$, where A is the matrix of (3.1). To bound $\| A^{-1} \|_\infty$, we use the results of the previous section.

Lemma 3.1. *For any Δ, $0 < a_{i+1} c_i < \frac{1}{4}$, $1 \leqslant i \leqslant N-1$.*

Proof.

$$\frac{1}{a_{i+1}c_i} = \frac{d_i d_{i+1}}{(1-\lambda_{i+1})\, h_i \lambda_i \lambda_{i+1}(1-\lambda_{i+2})\, h_{i+2}}.$$

Now,

$$\frac{d_i}{(1-\lambda_{i+1})\, h_i\lambda_i} > \frac{2-\lambda_{i+1}}{1-\lambda_{i+1}} > 2 \qquad \text{since } \lambda_{i+1} > 0.$$

Also,

$$\frac{d_{i+1}}{\lambda_{i+1}(1-\lambda_{i+2})\, h_{i+2}} > \frac{1+\lambda_{i+1}}{\lambda_{i+1}} > 2 \qquad \text{since } \lambda_{i+1} < 1.$$

Theorem 3.2. *Assume that* $| \lambda_i - \frac{1}{2} | \leqslant \gamma < \frac{1}{2}(2^{1/2} - 1)\ \forall i$. Then, there is a constant $K = K(\gamma)$ *independent of* Δ *and* $\bar{\lambda}$, such that $\| P(\bar{\lambda}, \Delta)\|_\infty \leqslant K$.

Proof. A routine diagonal dominance argument works (cf. [3]).

It can be shown that the above range of λ_i is sharp. Let $2^{1/2}/2 < \lambda < 1$ and let $\lambda_i = \lambda\ \forall i$. In this case

$$a_i = \lambda^2 h_{i+1} d_i^{-1} = \frac{\lambda^2}{1 - \lambda^2 + (h_i/h_{i+1})(2\lambda - \lambda^2)}.$$

Let $k_1 > 0$ be given. Since $\lambda > 1/2^{1/2}$, we may recursively choose $\{k_i\}_{i=2}^N$ so that

$$\frac{\lambda^2}{1 - \lambda^2 + \dfrac{k_i}{k_{i+1}}(2\lambda - \lambda^2)} > 1.$$

Let $y_1 = k_1$ and $y_{i+1} = y_i + k_i$ for $i = 1, 2,..., N-1$. Let $p(x) = (1/y_N)\,x$ and $x_i = p(y_i)$. Then, for the partition $\Delta: 0 = x_0 < \cdots < x_N = 1$, the a_i's of (3.1) are all greater than 1. Therefore, by (2.2) the inverse of the matrix A of (3.1) is bounded below by $N - 1$. That is, there is a vector $g = (g_1, ..., g_{N-1})^T$ with $\| g \|_\infty = 1$ so that $\| A^{-1} g \|_\infty \geqslant N - 1$. We still must show that the components of such a vector may be taken to be of the form

$$g_i = \frac{h_{i+1} f_i + h_i f_{i+1}}{h_{i+1}(1 - \lambda^2) + h_i(2\lambda - \lambda^2)}, \qquad 1 \leqslant i \leqslant N - 1,$$

where each $| f_i | \leqslant K$, for some constant K independent of N. Consider the matrix $B = (b_{ij})$ defined by $b_{ii} = h_{i+1}/d_i$, $b_{i,i+1} = h_i/d_i$, $b_{ij} = 0$ otherwise. We may assume that $h_i/h_{i+1} \leqslant \frac{1}{2}$ for all i. It suffices to show that $\| B^{-1} \|_\infty \leqslant K$ for some K independent of N. Now, B is strictly diagonally dominant so $\| B^{-1} \|_\infty \leqslant (\min_i \{b_{ii} - b_{i,i+1}\})^{-1} \leqslant (\min_i \{\frac{1}{2} h_{i+1} d_i^{-1}\})^{-1} \leqslant 2 + 2\lambda - 3\lambda^2$. In the case that $\lambda_i = 2^{1/2}/2$ for all i, the above argument can be modified to produce a_i's satisfying $g_i > 1 - \epsilon$ for any $\epsilon > 0$. In particular, one chooses $\{k_i\}$ so that $k_{i+1} > \epsilon^{-1} k_i (2(2)^{1/2} - 1)$.

THEOREM 3.3. *For every positive integer N and every* $\lambda \in (2^{1/2}/2, 1)$, *there is a partition* $\Delta: 0 = x_0 < \cdots < x_N = 1$ and a constant $K = K(\lambda)$ depending only on λ such that $\| P(\bar{\lambda}, \Delta)\|_\infty \geqslant K(\lambda)\, N$, *where* $\bar{\lambda} = (\lambda_1, ..., \lambda_N)$, $\lambda_i = \lambda$ *for all i.*

Proof. The computations above show that we may find a function f such that $\| f \|_\infty \leqslant 2 + 2\lambda - 3\lambda^2$ and $\| P(\bar{\lambda}, \Delta) f \|_\infty \geqslant N - 1$.

There are several ways in which the above results may be extended. By restricting the class of partitions under consideration to $\mathscr{P}_\sigma$ for some fixed $\sigma \geqslant 1$, one may hope to show that there is a constant $C(\sigma)$ so that if $| \lambda_i - \frac{1}{2} | \leqslant \gamma < C(\sigma)$ for all i, then $\| P(\bar{\lambda}, \Delta)\|_\infty \leqslant K(\gamma, \sigma)$ for some constant $K(\gamma, \sigma)$ independent of Δ and $\bar{\lambda}$. This is the case, at least for $\bar{\lambda}$ of the form $(\lambda, \lambda, ..., \lambda)$.

THEOREM 3.4. *Let* $\sigma \geqslant 1$ *be given. Let* $2\sigma - (1 + \sigma + 2\sigma^2)^{1/2} < \lambda(2\sigma + 1) < 1 + (1 + \sigma + 2\sigma^2)^{1/2}$. *Then, there exists a constant* $K(\sigma, \lambda)$ *so that for any* $\Delta \in \mathscr{P}_\sigma$, $\| P(\bar{\lambda}, \Delta)\|_\infty \leqslant K(\sigma, \lambda)$, *where* $\bar{\lambda} = (\lambda,..., \lambda)$. *Furthermore, these bounds on* λ *are best possible.*

Proof. In (3.1) let $h_{i+1} = \mu h_i$, then $a_i < 1$ if and only if $|(2\mu + 1)\lambda - 1| < (1 + \mu + 2\mu^2)^{1/2}$. Similarly, $c_i < 1$ if and only if $|(2 + \mu)\lambda - 2| < (\mu^2 + \mu + 2)^{1/2}$. The condition $\Delta \in \mathscr{P}_\sigma$ means that $\sigma^{-1} \leqslant \mu \leqslant \sigma$. One can check that the above inequalities hold for all μ in this range if and only if λ satisfies the hypotheses of this theorem. Also, if, for example, $h_{i+1} = \sigma^{-1}h_i$ for all i and $\lambda(2\sigma + 1) < 2\sigma - (1 + \sigma + 2\sigma^2)^{1/2}$, then $c_i > 1$ for all i. Therefore, the norm of matrix of (3.1) is bounded below by $N - 1$. An argument similar to that preceding Theorem 3.3 shows that $\| P(\bar{\lambda}, \Delta)\|_\infty \geqslant CN$ where C depends only on λ and σ. Q.E.D.

Another way of extending the results of this section is to restrict attention to narrower classes of functions then $C[0, 1]$ and to try to prove convergence results for arbitrary partitions. This is the topic of the next section.

4. APPROXIMATION OF SMOOTH FUNCTIONS

The modulus of continuity of a function $f \in C[0, 1]$ is defined to be $\omega(f, \delta) = \sup\{|f(x) - f(y)| : |x - y| \leqslant \delta\}$. We say $f \in \mathrm{Lip}_\alpha$ if there is a constant $M > 0$ so that $\omega(f, \delta) \leqslant M\delta^\alpha \ \forall\, \delta > 0$. Let $\bar{\lambda} = (\lambda_1, ..., \lambda_N)$ and let $P(\bar{\lambda}, \Delta)$ be as in the previous section. If $Pf = s$, then on $[x_{i-1}, x_i]$, s has the form

$$s(x) = \tfrac{1}{2}h_i^{-1}(s_i' - s_{i-1}')(x - \xi_i)^2 + (\lambda_i s_{i-1}' + (1 - \lambda_i)\, s_i')(x - \xi_i) + f(\xi_i), \tag{4.1}$$

where $s_i' \equiv s'(x_i)$ and $\xi_i = \lambda_i x_{i-1} + (1 - \lambda_i)\, x_i$. The constraint $s(x_i+) = s(x_i-)$ yields the equations for $1 \leqslant i \leqslant N - 1$

$$\alpha_i s_{i-1}' + (\alpha_i + \beta_i + \gamma_i)\, s_i' + \beta_i s_{i+1}' = 2(h_i + h_{i+1})^{-1}\{f(\xi_{i+1}) - f(\xi_i)\}, \tag{4.2}$$

where $\alpha_i = \lambda_i^2 h_i(h_i + h_{i+1})^{-1}$, $\beta_i = (1 - \lambda_{i+1})^2\, h_{i+1}(h_i + h_{i+1})^{-1}$ and $\gamma_i = (h_i + h_{i+1})^{-1}\{(1 - \lambda_i)\,\lambda_i h_i + (1 - \lambda_{i+1})\,\lambda_{i+1}h_{i+1}\}$.

Now, $s_0' = 2(1 - \lambda_1^2)^{-1}\, h_1^{-1}\{f(\xi_1) - f(0)\} - (1 + \lambda_1)^{-1}\,(1 - \lambda_1)\, s_1'$ and $s_N' = 2(2 - \lambda_N)^{-1}\{\lambda_N^{-1}h_N^{-1}[f(1) - f(\xi_N)] - \tfrac{1}{2}\lambda_N s_{N-1}'\}$. Therefore, for $i = 1, N - 1$ (4.2) becomes

$$\begin{aligned}(2(1 + \lambda_1)^{-1}\,\lambda_1\alpha_1 + \beta_1 + \gamma_1)\, s_1' + \beta_1 s_2'\\ = 2(h_1 + h_2)^{-1}\{f(\xi_2) - f(\xi_1) - \lambda_1^2(1 - \lambda_1)^{-2}\,[f(\xi_1) - f'0)]\},\end{aligned} \tag{4.3}$$

$$(\alpha_{N-1} + \gamma_{N-1} + 2(1 - \lambda_N)(2 - \lambda_N)^{-1} \beta_{N-1}) s'_{N-1} + \alpha_{N-1} s'_{N-2}$$
$$= 2(h_N + h_{N-1})^{-1} \{f(\xi_N) - f(\xi_{N-1})$$
$$- \lambda_N^{-1}(2 - \lambda_N)^{-1} (1 - \lambda_N)^2 [f(1) - f(\xi_N)]\}.$$

It is easily checked that the system (4.2)–(4.3) is strictly diagonally dominant and that the strength of the diagonal dominance depends only on the quantity $\min_i \{\lambda_i, 1 - \lambda_i\}$.

Assume now that $f \in \mathrm{Lip}_1$. Then, the right side of (4.2)–(4.3) is bounded by a constant K depending only on λ_1 and λ_N. The strict diagonal dominance of the system yields that

$$\max_i | s'_i | \leqslant M \max_i \{\gamma_i^{-1}\}. \tag{4.4}$$

Since $2\gamma_i \geqslant \min \{\lambda_i(1 - \lambda_i), \lambda_{i+1}(1 - \lambda_{i+1})$, it follows that

$$\sup_{0 \leqslant x \leqslant 1} \| s' \|_\infty = \max_i | s'_i | \leqslant 2M \max_i \{[\lambda_i(1 - \lambda_i)]^{-1}\}. \tag{4.5}$$

Note that the above bound is *independent of Δ*. Now, let $x \in [x_{i-1}, x_i]$, then $| s(x) - f(x)| \leqslant | s(x) - s(\xi_i)| + | f(\xi_i) - f(x)| \leqslant K | x - \xi_i | \max_i \{[\lambda_i (1 - \lambda_i)]^{-1}\}$.

We have proven:

THEOREM 4.1. *Let $f \in \mathrm{Lip}_1$. Assume there is an $\epsilon > 0$ so that $\epsilon \leqslant \lambda_i \leqslant 1 - \epsilon$ for all λ_i. Then, there is a constant K depending only on ϵ and the Lipschitz constant of f so that*

$$\| P(\lambda, \Delta) f - f \|_\infty \leqslant K \bar{\Delta}.$$

The above result shows that by assuming a little smoothness on the part of the functions we interpolate we may significantly improve the results of the previous section, in particular, Theorem 3.2. This phenomenon is not rare in spline theory; for example, cubic spline interpolation at the partition points with appropriate end conditions might not converge for a given continuous function; however, it does converge, and at the right rate, for all W_2^2 functions, cf. [5, 7].

Now, let $f \in \mathrm{Lip}_\alpha$ for some $0 < \alpha < 1$ and let $x \in [x_{i-1}, x_i]$. Then

$$| f(x) - s(x)| \leqslant | f(x) - f(\xi_i)| + | s(\xi_i) - s(x)|$$
$$\leqslant K h_i^\alpha + | s'_i || p_i(x)| + | s'_{i-1} || q_i(x)|, \tag{4.6}$$

where p_i and q_i can be determined from (4.1). We note that $|p_i(x)| \leqslant 2h_i$ and $|q_i(x)| \leqslant 2h_i$ for all $x_{i-1} \leqslant x \leqslant x_i$. Now,

$$s_i' = \sum_j \alpha_{ij} I_j(f) \tag{4.7}$$

where (α_{ij}) is the inverse of (4.2)–(4.3) and the $I_j(f)$'s are the components of the right-hand side of (4.2)–(4.3), e.g., $I_j(f) = 2(h_j + h_{j+1})^{-1} [f(\xi_{j+1}) - f(\xi_j)]$ for $2 \leqslant j \leqslant N-2$. If we assume that, for some $\epsilon > 0$, $\epsilon \leqslant \lambda_i \leqslant 1 - \epsilon$ for all i and that $\Delta \in \mathscr{P}_\sigma$ for some $\sigma \geqslant 1$, then we can bound (4.6) as follows

$$|s_i'| \leqslant K(h_i + h_{i+1})^{\alpha-1} \sum_j |\alpha_{ij}| (\sigma^{1-\alpha})^{|i-j|}, \tag{4.8}$$

where K depends on ϵ and f but not on Δ. Since the strength of the diagonal dominance of (4.2)–(4.3) does not depend on Δ, there are $K > 0$ and $0 < r < 1$ depending only on ϵ such that $|\alpha_{ij}| \leqslant Kr^{|i-j|}$. Therefore, if $\sigma^{\alpha-1} < r$, the right-hand side of (4.8) becomes a geometric series and yields the bound

$$|f(x) - s(x)| \leqslant Kh_i^\alpha, \qquad x_{i-1} \leqslant x \leqslant x_i, \tag{4.9}$$

where K depends on f, ϵ, and σ but not on Δ. We summarize this in

THEOREM 4.3. *Let $f \in \mathrm{Lip}_\alpha$. Assume there is an $\epsilon > 0$ so that $\epsilon \leqslant \lambda_i \leqslant 1 - \epsilon$ for all i. There is a constant $\sigma_0 = \sigma_0(\epsilon, \alpha) > 1$ such that if $\Delta \in \mathscr{P}_\sigma$ for some $\sigma < \sigma_0$, then there is a $K = K(f, \sigma, \epsilon)$ such that for $x_{i-1} \leqslant x \leqslant x_i$*

$$|P(\bar{\lambda}, \Delta) f(x) - f(x)| \leqslant Kh_i^\alpha.$$

This result may be viewed as an attempt to fill the gap between Theorems 3.2 and 4.1.

Remarks. 1. If the interpolation parameters λ_i are all equal to 0, it may not be the case that $P(\bar{\lambda}, \Delta) f \to f$ even for $f \in C^2[0, 1]$. A thorough analysis of this case can be found in [1].

2. All of the convergence results presented here are "local" in the sense of [3, p. 247]. This can be seen from the exponential decay that characterizes the elements of the matrices arising in interpolation. It is also a direct consequence of [2].

3. The results of this paper are no guarantee that "haphazard" interpolation by quadratic splines will actually produce good results numerically. While it is true that in every convergence proof presented the condition number, $\|A\|_\infty \|A^{-1}\|_\infty$, of the matrix to be inverted was independent of the size of the matrix, it is also true that this condition number may be very large to begin with.

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